Oscillation and Instabilities of Relativistic Stars ‡

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Abstract. In this short review we discuss the relevance of ongoing research into stellar oscillations and associated instabilities for the detection of gravitational waves and the future field of "gravitational-wave astronomy".

1. Introduction

As we enter the new millennium there is a focused worldwide effort to construct devices that will enable the first undisputed detection of gravitational waves. A network of large-scale ground-based laser-interferometer detectors (LIGO, VIRGO, GEO600, TAMA300) is due to come on-line soon, and the sensitivity of the several resonant mass detectors that are already in operation continues to be improved. An integral part in this effort is played by theoretical modeling of the expected sources. Theorists are presently racking their brains to think of various sources of gravitational waves that may be observable once the new ultra-sensitive detectors operate at their optimum level, and of any piece of information one may be able to extract from such observations.

The theory of stellar pulsation is richly endowed with interesting phenomena, and ever improving observations suggest that most stars exhibit complicated modes of oscillation. Thus it is natural to try to match theoretical models to observed data in order to extract information about the dynamics of distant stars. This interplay between observations and stellar pulsation theory is known as asteroseismology.

One of the most challenging goals that can (at least, in principle) be achieved via gravitational-wave detection is the determination of the equation of state of matter at supranuclear densities. We have recently argued that observed gravitational waves from the various nonradial pulsation modes of a neutron star can be used to infer both the mass and the radius of the star with surprisingly good accuracy, and thus put useful constraints on the equation of state [1, 2, 3, 4, 5]. But it is not clear that astrophysical mechanisms can excite the various oscillation modes to a detectable level. It seems likely that only the most violent processes, such as the actual formation of a neutron star following a supernova or a dramatic starquake following, for example, an internal phase-transition, will be of relevance. The strengthening evidence for magnetars [6], in which a starquake could release large amounts of energy, is also very interesting in

this respect. One can estimate that these events must take place in our immediate neighborhood (the Milky Way or the Local Group) in order to be observable.

The spectrum of a pulsating relativistic star is known to be tremendously rich, but most of the associated pulsation modes are of little relevance for gravitational-wave detection. From the gravitational-wave point of view one would expect the most important modes (in addition to the unstable r-modes [7]) to be the fundamental (f) mode of fluid oscillation, the first few pressure (p) modes and the first gravitational-wave (w) modes [8]. For more details on the theory of relativistic stellar pulsation we refer the reader to two recent review articles [9, 10]. That the bulk of the energy from an oscillating neutron star is, indeed, radiated through these modes has been demonstrated by numerical experiments [11, 12, 13, 14, 15, 16, 17].

Consequently it is relevant to try to device a strategy for detecting gravitational waves from pulsating neutron stars, and see what information such a detection could provide regarding the stars parameters. Such a strategy can potentially be of great importance to gravitational-wave astronomy, since most stars are expected to oscillate nonradially. In principle, one would expect the modes of a star to be excited in any dynamical scenario that leads to significant asymmetries. Of course, one can only hope to observe gravitational waves from the most compact stars (neutron stars and possibly strange stars) and only the most violent processes are of interest. Still, there are several scenarios in which the various pulsation modes may be excited to an interesting level: (a) A supernova explosion is expected to form a wildly pulsating neutron star that emits gravitational waves. The current estimates for the energy radiated as gravitational waves from supernovae is rather pessimistic, suggesting a total release of the equivalent to $10^{-6}M_\odot c^2$, or so. However, this may be a serious underestimate if the gravitational collapse in which the neutron star is formed is strongly non-spherical. Optimistic estimates suggest that as much as $10^{-2}M_\odot c^2$ may be released in extreme events. (b) Another potential excitation mechanism for stellar pulsation is a starquake, e.g., associated with a pulsar glitch. The typical energy released in this process may be of the order of the maximum mechanical energy that can be stored in the crust, estimated to be at the level of $10^{-9} - 10^{-7}M_\odot c^2$ [18, 19]. This is also an interesting possibility considering the recent suggestion that the soft-gamma repeaters are magnetars, neutron stars with extreme magnetic fields [20], that undergo frequent starquakes. It seems very likely that some pulsation modes are excited by the rather dramatic events that lead to the most energetic bursts seen from these systems. Indeed, Duncan [6] has recently argued that toroidal modes in the crust should be excited. If modes are excited in these systems, an indication of the energy released in the most powerful bursts is the $10^{-9}M_\odot c^2$ estimated for the March 5 1979 burst in SGR 0526-66. The maximum energy should certainly not exceed the total supply in the magnetic field $\sim 10^{-6}(B/10^{15}G)^2M_\odot c^2$ [20]. The possibility that a burst from a soft gamma-ray repeater may have a gravitational-wave analogue is very exciting. (c) The coalescence of two neutron stars at the end of binary inspiral may form a pulsating remnant. It is, of course, most likely, that a black hole is formed when two neutron stars coalesce, but even in that case the eventual collapse may be halted long enough (many dynamical timescales) that several oscillation modes could potentially be identified [21]. Also, stellar oscillations can be excited by the tidal fields of the two stars during the inspiral phase that precedes the merger [22]. (d) The star may undergo a dramatic phase-transition that leads to a mini-collapse. This would be the result of a sudden softening of the equation of state (for example, associated with the formation of a condensate consisting of pions or kaons). A phase-

transition could lead to a sudden contraction during which a considerable part of the stars gravitational binding energy would be released, and it seems inevitable that part of this energy would be channeled into pulsations of the remnant. Large amounts of energy that could be released in the most extreme of these scenarios: a contraction of (say) 10% can easily lead to the release of $10^{-2}M_\odot c^2$. Transformation of a neutron star into a strange star is likely to induce pulsations in a similar fashion. It is reasonable to assume that the bulk of the total energy of the oscillation is released through a few of the stars quadrupole pulsation modes in all of these scenarios. We will assume that this is the case and assess the likelihood that the associated gravitational waves will be detected. Having done this we discuss the inverse problem, and investigate how accurately the neutron star parameters can be inferred from the gravitational wave data. Finally, we briefly discuss the gravitational-wave driven instability of the so-called r-modes.

2.   Nonradial stellar oscillations: Theoretical minimum

A neutron star has a large number of families of pulsation modes with more or less distinct character. For the simplest stellar models, the relevant modes are high frequency pressure p-modes and the low frequency gravity g-modes [23]. For a typical nonrotating neutron star model the fundamental p-mode (usually referred to as the f-mode), whose eigenfunction has no nodes in the star, has frequency in the range 2-4 kHz, while the first overtone lies above 4 kHz. The g-modes depend sensitively on the internal composition and temperature distribution, but they typically have frequencies of a few hundred Hz. The standard mode-classification dates back to the seminal work of Cowling [24], and is based on identifying the main restoring force that influences the fluid motion. As the stellar model is made more detailed and further restoring forces are included new families of modes come into play. For example, a neutron star model with a sizeable solid crust separating a thin ocean from a central fluid region will have g-modes associated with both the core and the ocean as well as modes associated with shearing motion in the crust[25, 26]. Of particular interest to relativists is the existence of a class of modes uniquely associated with the spacetime itself [27, 28, 8, 29]; the so-called w-modes (for gravitational <u>w</u>ave). These modes essentially arise because the curvature of spacetime that is generated by the background density distribution can temporarily trap impinging gravitational waves. The w-modes typically have high frequencies (above 6 kHz) and damp out in a fraction of a millisecond. It is not yet clear whether one should expect these modes to be excited to an appreciable level during (say) a gravitational collapse following a supernova. One might argue that they provide a natural channel for the release of any initial deformation of the spacetime, but there are as yet no solid evidence indicating a significant level of w-mode excitation in a realistic scenario [11, 12, 10, 14, 15, 16, 17].

3.   Gravitational Wave Asteroseismology

Once gravitational waves are detected the first task will be to identify the source. This should be possible from the general character of the waveform and may not require very accurate theoretical models, but such models will be of crucial importance for a deduction of the parameters of the source. That is, for gravitational-wave "astronomy".

   The idea behind this presentation is a familiar one in astronomy: For many

years, studies of the light variation of variable stars have been used to deduce their internal structure [23]. The Newtonian theory of stellar pulsation was to a large extent developed in order to explain the pulsations of Cepheids and RR Lyrae. This approach, known as Asteroseismology (Helioseismology in the specific case of the Sun), has been remarkably successful in recent years. In comparison, the relativistic theory of stellar pulsation, which has now been developed for thirty years, has not yet been applied in a similar way. So far, the relativistic theory has no immediate connections to observations (that are not already provided by the Newtonian theory). We believe that this situation will change once the gravitational-wave window to the universe is opened, and in this review we discuss how the information carried by the gravitational-wave signal can be inverted to estimate the parameters of pulsating stars. That is, we take a first step towards gravitational-wave asteroseismology.

3.1.   What can we learn from observations?

Our present understanding of neutron stars comes mainly from X-ray and radio-timing observations. These observations provide some insight into the structure of these objects and the properties of supranuclear matter. The most commonly and accurately observed parameter is the rotation period, and we know that radio pulsars can spin very fast (the shortest observed period being the 1.56 ms of PSR 1937+21). Another basic observable, that can be obtained (in a few cases) with some accuracy from todays observations, is the mass of the neutron star. As Finn [30] has shown, the observations of radio pulsars indicate that the mass lies in the range $1.01 < M/M_\odot < 1.64$. Similarly, van der Kerkwijk et al [31] find that data for X-ray pulsars indicate that $1.04 < M/M_\odot < 1.88$. The data used in these two studies is actually consistent with (if one includes error bars) $M < 1.44 M_\odot$. We now recall that the various EOS that have been proposed by theoretical physicists can be divided into two major categories: i) the "soft" EOS which typically lead to neutron star models with maximum masses around $1.4 M_\odot$ and radii usually smaller than 10 km, and ii) the "stiff" EOS with maximum values $M \sim 1.8 M_\odot$ and $R \sim 15$ km [32]. We thus see that, even though the constraint put on the neutron star mass by present observations seems strong, it actually does not rule out many of the proposed EOS. In order to arrive at a more useful result we are likely to need detailed observations also of the stellar radius. Unfortunately, available data provide little information about the radius. The recent observations of quasiperiodic oscillations in low mass X-ray binaries indicate that $R < 6M$, but again this is not a severe constraint. Although a number of attempts have been made, using either X-ray observations [33] or the limiting spin period of neutron stars [34], to put constraints on the mass-radius relation, we do not yet have a method which can provide the desired answer. In view of this situation, any method that can be used to infer neutron star parameters is a welcome addition. Of specific interest may be the new possibilities offered once gravitational-wave observations become reality.

Let us suppose that a nearby supernova explodes, say in the Local Group of galaxies, and is followed by a core collapse that leads to the formation of a compact object. As the dust from the collapse settles the compact object pulsates wildly in its various oscillation modes, generating a gravitational-wave signal which is composed of an overlapping of different frequencies. We will assume that the results of Allen et al. [11] can be brought to bear on this situation, i.e. that most of the energy is radiated through the f-mode, a few p-modes and the first w-mode. Our detector picks up this signal, and a subsequent Fourier analysis of the data stream yields the frequencies and

the energy carried by each mode.

The first question to be answered by the gravitational-wave astronomer concerns what kind of compact object could produce the detected signal. Is it a black hole or a neutron star? The pulsation of these objects lead to qualitatively similar gravitational waves, eg. exponentially damped oscillations, but the question should nevertheless be relatively easy to answer. If more than one of the stellar pulsation modes is observed the answer is clear, but even if we only observe only one single mode the two cases should be easy to distinguish. The fundamental (quadrupole) quasinormal mode frequency of a Schwarzschild black hole follows from

$$f \approx 12\text{kHz} \left(\frac{M_\odot}{M} \right) , \tag{1}$$

while the associated e-folding time is

$$\tau \approx 0.05\text{ms} \left(\frac{M}{M_\odot} \right) . \tag{2}$$

That is, the oscillations of a 10 $M_\odot$ black hole lie in the frequency range of the f-mode for a typical neutron star. But the two signals will differ greatly in the damping time, the e-folding time of the black hole being nearly three orders of magnitude shorter than that of the neutron star f-mode.

Having excluded the possibility that our signal came from a black hole, we want to know the mass and the radius of the newly born neutron star. We also want to decide which of the proposed EOS that best represents this star. To address these questions we can use a set of empirical relations that can be used to estimate the mass, the radius and the EOS of the neutron star with good precision.

3.2. Addressing the inverse problem

Considering the possibility of a future detection it is relevant to pose the "inverse problem" for gravitational waves from pulsating stars. Once we have observed the waves, can we deduce the details of the star from which they originated? To answer this question we have calculated the frequencies and damping times of the modes that we expect to lead to the strongest gravitational waves for a selection of EOS. Nearly twenty years ago Lindblom and Detweiler [35] tabulated the frequencies and damping times of the f-mode for a number of EOS. Recently we [2] extended their calculation by adding more recent EOS and providing data also for the w- and p- modes. These numerical data were then used to to create useful "empirical" relations between the "observables" (frequencies and damping times) and the parameters of the star (mass, radius and possibly the EOS). We will now outline how these relations can be used to infer the stellar parameters from detected mode data.

Let us first consider the frequency of the f-mode. It is well known that the characteristic time-scale of any dynamical process is related to the mean density ($\bar{\rho}$) of the mass involved. This notion should be relevant for the fluid oscillation modes of a star, and we consequently expect that $\omega_f \sim \bar{\rho}^{1/2}$. That is, we should normalize the f-mode frequency with the average density of the star. As shown in [2] the relation between the f-mode frequencies and the mean density is almost linear, and a linear fitting leads to the following simple relation:

$$\omega_f(\text{kHz}) \approx 0.78 + 1.635 \left(\frac{\bar{M}}{\bar{R}^3} \right)^{1/2} , \tag{3}$$

where we have introduced the dimensionless variables

$$\bar{M} = \frac{M}{1.4 M_\odot} \quad \text{and} \quad \bar{R} = \frac{R}{10\text{km}} \ . \tag{4}$$

From equation (3) follows that the typical f-mode frequency is around 2.4 kHz.

To deduce a similar relation for the damping rate of the f-mode, we can use the rough estimate given by the quadrupole formula. That is, the damping time should follow from

$$\tau_f \sim \frac{\text{oscillation energy}}{\text{power emitted in GWs}} \sim R \left(\frac{R}{M} \right)^3 \ . \tag{5}$$

Using this relation and the numerical results for the damping time of the f-mode for various stellar parameters M and R, we find that [2]:

$$\frac{1}{\tau_f(\text{s})} \approx \frac{\bar{M}^3}{\bar{R}^4} \left[22.85 - 14.65 \left(\frac{\bar{M}}{\bar{R}} \right) \right] \tag{6}$$

and it follows that the typical damping time of the f-mode is around 0.15 sec. Analogous empirical relations for the p-mode data are much less robust and useful. This is natural since the p-modes are sensitive to changes in the matter distribution inside the star (as manifested through changes in the sound speed). In contrast, the gravitational-wave w-modes lead to very robust results. It is well known [8, 29] that the w-modes do not excite a significant fluid motion. Thus, they are more or less independent of the characteristics of the fluid: The frequencies do not depend on the sound speed and the damping times cannot be modeled by the quadrupole formula. Analytic results for model problems for the w-modes [27, 36], show that the frequency of the w-mode is inversely proportional to the size of the star. Meanwhile, the damping time is related to the compactness of the star, i.e. the more relativistic the star is the longer the w-mode oscillation lasts. These properties have already been discussed in some detail in [29] for uniform density stars. One can find the following relations for the frequency and damping of the first w-mode:

$$\omega_w(\text{kHz}) \approx \frac{1}{R} \left[20.92 - 9.14 \left(\frac{\bar{M}}{\bar{R}} \right) \right] \ , \tag{7}$$

and

$$\frac{1}{\tau_w(\text{ms})} \approx \frac{1}{\bar{M}} \left[5.74 + 103 \left(\frac{\bar{M}}{\bar{R}} \right) - 67.45 \left(\frac{\bar{M}}{\bar{R}} \right)^2 \right] \ . \tag{8}$$

We see that a typical value for the w-mode frequency is 11-12 kHz, but since the frequency depends strongly on the radius of the star it varies greatly for different EOS. For example, for a very stiff EOS (L) the w-mode frequency is around 6 kHz while for the softest EOS in our set (G) the typical frequency is around 14 kHz. The w-mode damping time is comparable to that of an oscillating black hole with the same mass, i.e. it is typically less than a tenth of a millisecond.

3.3. Noisy signals

Suppose that one tries to detect the gravitational waves associated with the stellar pulsation modes that are excited when (say) a neutron star forms after a supernova

explosion. Since all modes are relatively short lived, the detection situation is similar to that for a perturbed rotating black hole [37, 38]. For each individual mode the signal is expected to have the following form:

$$h(t) = \begin{cases} 0 & \text{for } t < T, \\ \mathcal{A}e^{-(t-T)/\tau}\sin[2\pi f(t-T)] & \text{for } t \geq T. \end{cases} \tag{9}$$

Here, $\mathcal{A}$ is the initial amplitude of the signal, T is its arrival time, and f and τ are the frequency and damping time of the oscillation, respectively. Since the violent formation of a neutron star is a very complicated event, the above form of the waves becomes realistic only at the late stages when the remnant is settling down and its pulsations can be accurately described as a superposition of the various modes, either fluid or spacetime ones, that have been excited. At earlier times $(t < T)$ the waves are expected to have a random character that is completely uncorrelated with the intrinsic noise of an earth-bound detector. This partly justifies our simplification of setting the waveform equal to zero for $t < T$.

The energy flux F carried by any weak gravitational wave h is given by

$$F = \frac{c^3}{16\pi G}|\dot{h}|^2 \ , \tag{10}$$

where c is the speed of light and G is Newton's gravitational constant. Thus, when gravitational waves emitted from a pulsating neutron star hit such a detector on Earth, their initial amplitude will be [39]

$$\mathcal{A} \sim 2.4 \times 10^{-20} \left(\frac{E_{\text{gw}}}{10^{-6}M_\odot c^2}\right)^{1/2} \left(\frac{10\text{kpc}}{r}\right) \left(\frac{1\text{kHz}}{f}\right) \left(\frac{1\text{ms}}{\tau}\right)^{1/2} \ , \tag{11}$$

where E_{gw} is the energy released through the mode and r is the distance between detector and source. In order to dig out this kind of signal from the noisy output of a detector one could use templates of the same form as the expected signal (so called matched filtering). Following the analysis of Echeverria [37] the signal-to-noise ratio is found to be

$$\left(\frac{S}{N}\right)^2 = \rho^2 \equiv 2\langle h \mid h \rangle = \frac{4Q^2}{1+4Q^2}\frac{\mathcal{A}^2\tau}{2S_n} \ , \tag{12}$$

with

$$Q \equiv \pi f \tau \ , \tag{13}$$

being the quality factor of the oscillation, and S_n the spectral density of the detector (assumed to be constant over the bandwidth of the signal).

3.4. Are the modes detectable?

Two separate questions must be addressed in any discussion of gravitational-wave detection. The first one concerns identifying a weak signal in a noisy detector, thus establishing the presence of a gravitational wave in the data. The second question regards extracting the detailed parameters of the signal, e.g., the frequency and e-folding time of a pulsation mode. To address either of these issues we need an estimate of the spectral noise density S_n of the detector.
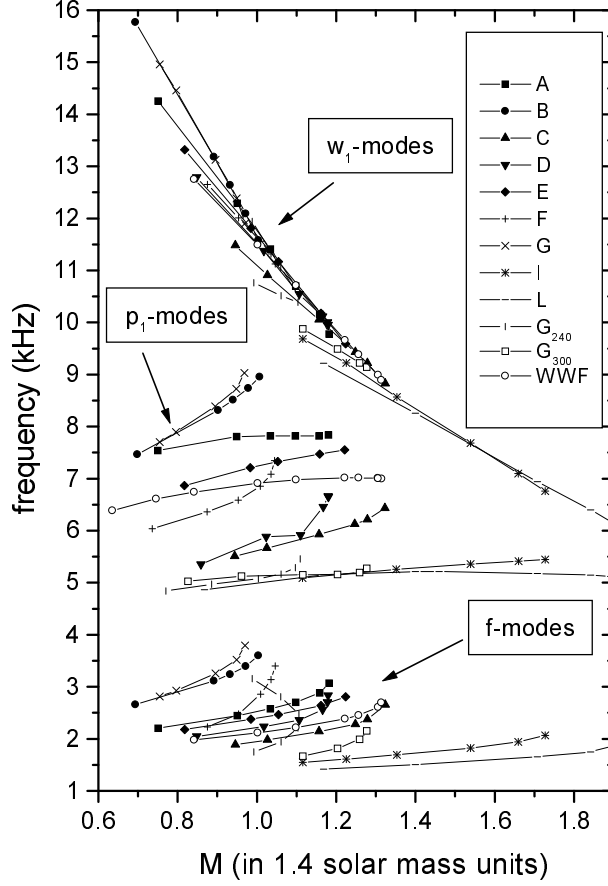
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Figure 1. This diagram shows the range of mode-frequencies for twelve different equations of state as functions of the normalized mass ($\bar{M} = M/1.4M_\odot$) of the star.

The pulsation modes of a (non-rotating) neutron star that may be detectable through the associated gravitational waves all have rather high frequencies; typically of the order of several kHz. To illustrate this we show the mode-frequencies for all models considered in [2] as a function of the stellar mass in Figure 1. From this figure we immediately see that a detector must be sensitive to frequencies of the order of 8-12 kHz and above to observe most w-modes. Of course, it is also clear that some equations of state yield w-modes with lower frequencies. For example, for massive neutron stars with $M \approx 1.8 - 2.3 M_\odot$ (stiff EOS), which may be needed to account for data for low-mass X-ray binaries, the w-mode frequency could be as low as 6 kHz (see also recent results for the axial w-modes [4]). The p-modes lie mainly in the

range 4-8 kHz, while all f-modes have frequencies lower than 4 kHz. This means that the mode-signals we consider lie in the regime where an interferometric detector is severely limited by the photon shot noise. For this reason a detection strategy based on resonant detectors (bars, spheres or even networks of small resonant detectors [40]) or laser interferometers operating in dual recycling mode, in order to overcome the overwhelming photon shot noise at high frequencies [41], seems the most promising. In fact, the range of mode-frequencies in Figure 1 provides strong motivation for detailed studies into the prospects for construction of dedicated ultrahigh frequency detectors.

In the following we will compare three different detectors: the initial and advanced LIGO interferometers, for which

$$S_n^{1/2} \approx h_m \left(\frac{f}{\alpha f_m} \right)^{3/2} \frac{1}{\sqrt{f}} \; \mathrm{Hz}^{-1/2} \; , \tag{14}$$

with $h_m = 3.1 \times 10^{-22}$, $\alpha = 1.4$ and $f_m = 160$ Hz for the initial configuration, and $h_m = 1.4 \times 10^{-23}$, $\alpha = 1.6$ and $f_m = 68$ Hz for the advanced configuration [42]. We also consider an "ideal" detector that is tuned to the frequency of the mode and has sensitivity of the order of $S_n^{1/2} \approx 10^{-24}$ Hz$^{-1/2}$. This is the sensitivity goal of the new generation of detectors under construction, see Figure 2. As an example of a suitably advanced instrument we will take the so called EURO detector, for which the noise-level curves have been estimated by Sathyaprakash and Schutz (private communication). It should be noted that the Advanced LIGO estimates are roughly valid also for spherical detectors such as TIGA, cf. Harry, Stevenson and Paik [43].

The detectability of the f, p and w-modes for different detectors can be assessed from (12). The main problem in doing this is the lack of realistic simulations providing information about the level of excitation of various modes in an astrophysical situation. Still, given the frequency and damping rate of a specific mode we can ask what amount of energy must be channeled through the mode in order for it to be detectable by a given detector. We immediately find that detection of pulsating neutron stars from outside our own galaxy is very unlikely. Let us consider a "typical" stellar model for which the f-mode has parameters $f_f = 2.2$ kHz and $\tau_f = 0.15$ s, this corresponds to a $1.4 M_\odot$ neutron star according to the Bethe-Johnson equation of state [2]. For this example we find that the f-mode in the Virgo cluster (at 15 Mpc) must carry an energy equivalent to more than $0.3 M_\odot c^2$ to lead to a signal-to-noise ratio of 10 in our ideal detector. Given that the total energy estimated to be radiated as gravitational waves in a supernova is at the level of $10^{-5} - 10^{-6} M_\odot c^2$, we cannot realistically expect to observe mode-signals from far beyond our own galaxy.

This means that the number of detectable events may be rather low. Certainly, one would not expect to see a supernova in our galaxy more often than once every thirty years or so. Still, there are a large number of neutron stars in our galaxy, all of which may be be involved in dramatic events (see the introduction for some possible scenarios) that lead to the excitation of pulsation modes. The energies required to make each mode detectable (with a signal-to-noise ratio of 10) from a source at the center of our galaxy (at 10 kpc) are listed in Table 1. In the table we have used the data for the "typical" stellar model, for which the characteristics of the f-mode were given above, $f_p = 6$ kHz and $\tau_p = 2$ s, and $f_w = 11$ kHz and $\tau_w = 0.02$ ms. This data indicates that, even though the event that excites the modes must be violent, the energy required to make each mode detectable is not at all unrealistic. In fact, the energy levels required for both the f- and p-modes are such that detection of violent

events in the life of a neutron star should be possible, given the Advanced LIGO detectors (or alternatively spheres with the sensitivity proposed for TIGA). On the other hand, detection of w-modes with the broad band configuration of LIGO seems unlikely. Detection of these modes, which would correspond to observing a purely relativistic phenomenon, requires dedicated high frequency detectors operating in the frequency range above 6 kHz. We believe that the data in Table 1 illustrates that neutron star pulsation modes may well be detectable from within our galaxy. The first detection may come as soon as the first generation of LIGO detectors come on line, but it may be more realistic to expect that we need a third generation detector (such as EURO) to truly probe the pulsation modes of neutron stars.

Detector	f-mode	p-mode	w-mode
LIGO I	4.9×10^{-5}	4.0×10^{-3}	6.8×10^{-2}
LIGO II	8.7×10^{-7}	7.0×10^{-5}	1.2×10^{-3}
Ideal	1.4×10^{-8}	1.3×10^{-7}	6.4×10^{-7}

Table 1. The estimated energy (in units of $M_\odot c^2$) required in each mode in order to lead to a detection with signal-to-noise ratio of 10 from a pulsating neutron star at the center of our galaxy (10 kpc), cf. Eqs (3,4). The given data corresponds to a $1.4 M_\odot$ star with the Bethe-Johnson equation of state.

3.5. How well can we determine the mode parameters?

Let us now discuss the precision with which we can hope to infer the details of each pulsation mode. We can compute the relative measurement error in the frequency and the damping time of the waves by some appropriately designed detector [3]. After introducing a convenient parameter $\mathcal{P}$, defined by

$$\mathcal{P}^{-1} = \left(\frac{S_n^{1/2}}{10^{-24} \mathrm{Hz}^{-1/2}} \right) \left(\frac{r}{10 \mathrm{kpc}} \right) \left(\frac{E_\mathrm{gw}}{10^{-6} M_\odot c^2} \right)^{-1/2} , \qquad (15)$$

we find that the error estimates take the following form

$$\frac{\sigma_f}{f} \simeq 0.0042 \, \mathcal{P}^{-1} \sqrt{\frac{1 - 2Q^2 + 8Q^4}{4Q^4}} \left(\frac{\tau}{1\mathrm{ms}} \right)^{-1} , \qquad (16)$$

and

$$\frac{\sigma_\tau}{\tau} \simeq 0.013 \, \mathcal{P}^{-1} \sqrt{\frac{10 + 8Q^2}{Q^2}} \left(\frac{f}{1\mathrm{kHz}} \right) . \qquad (17)$$

Also, for the time of arrival of the gravitational wave signal we get from

$$\sigma_T \simeq 0.0042 \, \mathcal{P}^{-1} \mathrm{ms} . \qquad (18)$$

To illustrate these results we list in Table 2 the relative errors associated with the parameter extraction for the "typical" $1.4M_\odot$ stellar model we used in the previous section. We assume that each mode carries the energy required for it to be observed with signal-to-noise ratio of 10, cf. Table 1. (This is a convenient measure since it is independent of the particulars of the detector.)
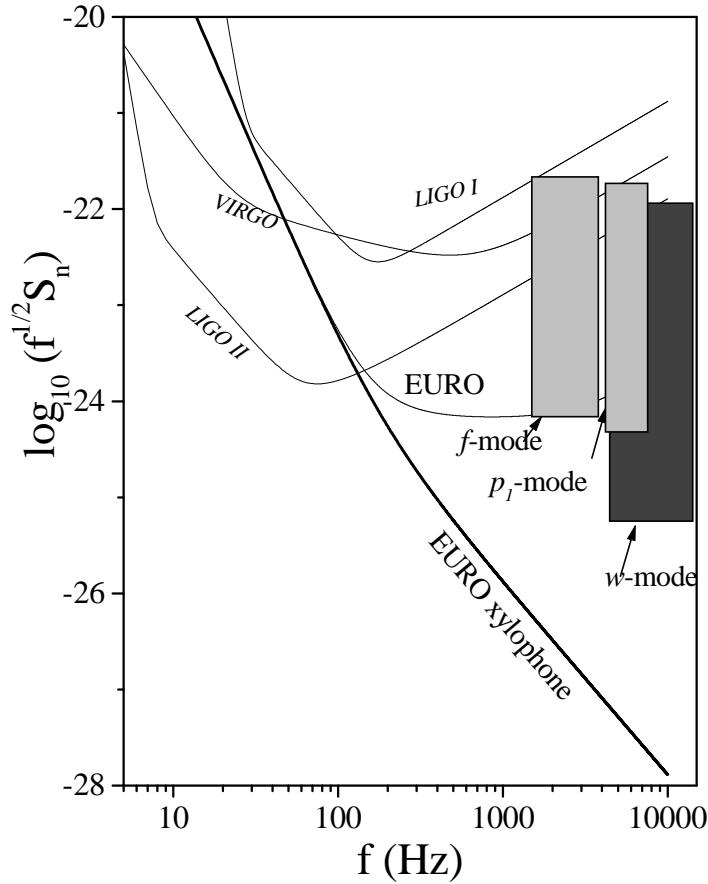

Figure 2. The spectral noise density for the new generation of laser interferometric gravitational wave detectors. EURO is a third generation detector with ideal characteristics for gravitational wave asteroseismology. The boxes represent the sensitivity needed to detect a mode into which an energy of $10^{-6}M\odot c^2$ is deposited. The upper limit of each box corresponds to an event in our galaxy and the lower in the Virgo cluster.

From the sample data in Table 2 one sees clearly that, while an extremely accurate determination of the frequencies of both the f- and the p-mode is possible, it would

Mode	σ_f/f	σ_τ/τ	σ_T (10^{-6} s)
f-mode	8×10^{-6}	0.02	1.1
p-mode	3×10^{-7}	0.02	0.38
w-mode	0.01	0.03	0.177

Table 2. The relative errors in the extraction of the mode parameters assuming a signal-to-noise ratio of 10. The relatively large error in estimating f_w is due to the rapid damping of these modes (small Q factor), cf. Eq. (12). In other words, we will not be able to detect more than a couple of cycles of a w-mode whereas many hundred cycles of both the f- and the p-modes could be observed.

be much harder to infer their respective damping rates. It is also clear that an accurate determination of both the w-mode frequency and damping will be difficult. To illustrate this result in a different way, we can ask how much energy must be channeled through each mode in order to lead to a 1% relative error in the frequency or the damping rate, respectively. Let us call the corresponding energies E_f and E_τ. This measure will then be detector dependent, so we list the relevant estimates for the three detector configurations used in Table 1. When the data is viewed in this way, cf. Table 3, we see that an accurate extraction of w-mode data will not be possible unless a large amount of energy is released through these modes. Furthermore, one would clearly need a detector that is sensitive at ultrahigh frequencies.

Detector	f-mode		p-mode		w-mode	
	E_f	E_τ	E_f	E_τ	E_f	E_τ
LIGO I	3×10^{-9}	2×10^{-2}	3×10^{-10}	—	—	—
LIGO II	5×10^{-11}	3×10^{-4}	5×10^{-12}	3×10^{-2}	0.2	—
Ideal	9×10^{-13}	6×10^{-6}	9×10^{-15}	5×10^{-5}	9×10^{-5}	6×10^{-4}

Table 3. The estimated energy (in units of $M_\odot c^2$) required in each mode in order to lead to a relative error of 1% in the inferred mode-frequency (E_f) and damping rate (E_τ). In cases where no entry is given, the required energy is unrealistically high (typically larger than $M_\odot c^2$). The distance to the source is assumed to be 10 kpc.

3.6. Revealing the position of the source

In the previous section we discussed issues regarding the detectability of a mode-signal, and the accuracy with which the parameters of the mode could be inferred from noisy gravitational wave data. Let us now assume that we have detected the mode and extracted the relevant parameters. We then naturally want to constrain the supranuclear equation of state by deducing the mass and the radius of the star. In principle, the mass and the radius can be deduced from any two observables, [2].

In the absence of detector noise, several combinations look promising, but in reality only few combinations are likely to be useful.

As with other kinds of gravitational-wave sources, a network of at least three detectors is needed to pinpoint the location of the source in the sky. The difference in arrival time for the three detectors could be used to determine the position of the source. The higher the accuracy in measuring the time of arrival at each detector, the more precise will be the positioning of the source. Two remote detectors, at a distance d apart from each other will receive the signal with a temporal difference of

$$\Delta T = \frac{d}{c} \cos \theta , \tag{19}$$

where c is the speed of light, and θ is the angle between the line joining the two detectors and the line of sight of the source. Therefore, the accuracy by which this angle can be measured is

$$\Delta \theta = \frac{\sqrt{2}\sigma_T c}{d \sin \theta} . \tag{20}$$

The $\sqrt{2}$ arises from the measurement errors of the two times of arrival. If one assumes an 'L' shaped network of 3 detectors with arm length of $d = 10,000$ km, Eqs. (18) and (20) lead to an error box on the sky with angular sides of 1°, at most (for specific areas of the sky, and large signal-to-noise ratios the angular sizes could be much smaller). This is quite interesting since one could then correlate the detection of gravitational waves with radio, X-ray or gamma-ray observations directed towards that specific corner of the sky.

4. The unstable r-modes

So far we have discussed the modes of (essentially) non-rotating stars. A more optimistic scenario is based on the notion that various modes of oscillation may be unstable in a rotating star. Following the serendipitous discovery of such an instability in the so-called r-modes [44, 45], this area of research has attracted considerable attention. Should such an instability operate in a young neutron star it may lead to the emission of copious amounts of gravitational waves [46]. Such gravitational waves have been estimated to be detectable for sources in the Virgo cluster (at 15-20 Mpc). If we suppose that most newly born neutron stars pass through a phase where this kind of instability is active, several such events should be observed per year once the advanced interferometers come into operation. This is a very exciting prospect, indeed.

In this review article we give a brief introduction to these recent ideas and suggestions. For an exhaustive discussion we refer the reader to [7].

The r-modes are unstable to the emission of gravitational waves via a mechanism that was first suggested by Chandrasekhar [47]. Subsequent work by Friedman and Schutz [48] showed that this instability is a generic feature of all rotating fluids, and in the case of r-modes Friedman and Morsink [45] showed that that the new instability is generic for toroidal perturbations of relativistic stars.

In a simple description, the instability works as follows: In a rapidly rotating star a backward moving mode (as measured by a co-rotating observer) can be dragged forward according to an inertial observer. This means that the mode radiates positive angular momentum, even though the angular momentum of the mode remains negative

because the perturbed star has lower net angular momentum than the unperturbed star. As positive angular momentum is removed from the mode, its angular momentum becomes increasingly negative, implying that its amplitude increases. The mode grows due to the emission of gravitation radiation. In other words, a mode is unstable if it is prograde relative to infinity and retrograde relative to the star. For many years the investigations into the relevance of the CFS instability was focussed on the spheroidal f-mode (for a recent review, see [49]). The reason for this is that this mode was considered the most important one as far as gravitational waves are concerned. Hence, it came as some surprise that the instability associated with the toroidal r-modes is considerably stronger than the f-mode one.

In fact, toroidal modes of relativistic stars attracted little attention until recently. Such perturbations of non-rotating stars lead to a set of zero frequency modes in Newtonian theory, complemented by gravitational-wave modes (w-modes) in the relativistic description [50, 51]. Toroidal modes of a Newtonian star describe stationary horizontal fluid currents, and do not induce variations in the density and pressure. When the star is set into rotation the Coriolis force provides a weak restoring force that gives the toroidal modes true dynamics, and the fluid undergoes oscillations with a frequency (measured by an inertial observer at infinity)

$$\sigma = -m\Omega + \frac{2m\Omega}{\ell(\ell+1)} \ , \tag{21}$$

where Ω is the rotational frequency of the star and ℓ and m are the spherical harmonic indices. This result, that identifies the r-modes, follows from a Newtonian treatment of oscillations of slowly rotating stars. As one can easily deduce from their frequency the r-modes are always retrograde in the corotating frame and their phase velocity is always smaller than that of the rotation of the star; thus they are generically unstable independently of the rotation rate of the star.

That a mode is formally unstable in a perfect fluid star does not in itself mean that it will be allowed to grow and affect, for example, the star's spin evolution. In any "realistic" star there are dissipation mechanisms that may halt growth of an instability. In the simplest model, one must account for the effects of bulk and shear viscosity. Comparison between the damping times due to viscosity and the growth time due to gravitational radiation provides a criterion for the significance of the instability. The damping/growth time associated with each mechanism can be estimated from the ratio of the energy loss to the available mode-energy (measured in the rotating frame) i.e.

$$\frac{1}{\tau_{\text{diss}}} = -\frac{\dot{E}_{\text{diss}}}{2E_{\text{mode}}} \tag{22}$$

In our case the onset of the instability is signalled by

$$\frac{1}{\tau} = -\frac{1}{\tau_{gw}} + \frac{1}{\tau_{sv}} + \frac{1}{\tau_{bv}} = 0. \tag{23}$$

In this relation, the growth time due to gravitational waves *(gw)* is temperature independent while both bulk *(bv)* and shear viscosity *(sv)* are strongly dependent on the internal temperature of the star. We can now find the critical rotation frequency at which the r-mode becomes unstable for the relevant values of the core temperature of the star. Detailed calculations have shown that bulk viscosity damps any fluid motion for temperatures higher than 10^{10} K, while shear viscosity dominates for temperatures

below 10^6 K. This means that there is a "window of opportunity" between $10^6 - 10^{10}$ K where the r-mode instability is active and may play an astrophysical role [52, 53]. This instability window is shown in Figure 3.

Finally, it is worth pointing out that the gravitational radiation emitted from the r-modes comes primarily from the time-dependent *mass-currents*. This is the gravitational analogue of magnetic multipole radiation and the r-mode instability is unique among expected astrophysical sources of gravitational radiation in radiating primarily by gravitomagnetic effects. The detectability of these gravitational waves is discussed in detail in [7].
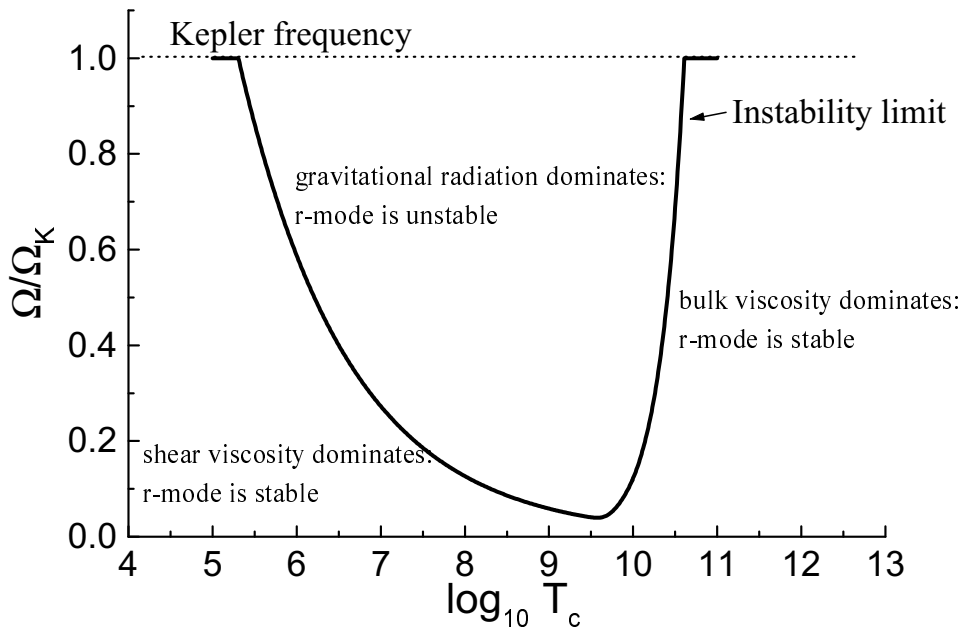

Figure 3. The critical rotation rate at which shear viscosity (at low temperatures) and bulk viscosity (at high temperatures) balance the r-mode instability

4.1. A novel scenario

Before we conclude this review, it is appropriate to comment on the fact that we neglected the effects of both rotation and magnetic fields on the pulsation modes in our discussion of asteroseismology. This was (obviously!) not because rotation and magnetic fields play an insignificant role. On the contrary, we expect rotation to be highly relevant in many cases (in particular since various instabilities may set in above a critical spin rate), and the recent evidence for magnetars make it clear that one cannot neglect the role of the magnetic field. Still, as far as most pulsation modes are concerned, one would expect rotation to have a significant effect only for neutron stars with very short period, and the present study may well be reasonable for stars

with periods longer than, say, 20 ms. Furthermore, it has been argued that neutron stars are typically born slowly rotating [54]. If that is the typical case, then our results could be relevant also for most newly born neutron stars. A more pragmatic reason for not including rotation in the present study is that detailed data for modes of rotating neutron stars is not yet available. Once such data has been computed the present study can be extended to incorporate rotational effects. There are currently efforts by various groups around the world to estimate the effects of rotation on f-, p- and w-modes (and also calculate the r-modes) in the framework of general relativity. There are already preliminary results for slowly-rotating stars [55, 44] but a complete study of both slowly and fast rotating stars is outstanding.

One of the main rotational effects is the splitting of the various frequencies, i.e. the frequency is no longer degenerate with respect to the harmonic index m. For example, the quadrupole f-mode oscillations ($l = 2$) will emit gravitational waves with all five possible values of m (-2,-1,0,1,2). This splitting of the spectrum will create significant problems in using the empirical relations such as (3). One has to derive a similar formula with an additional term of the form $m\Omega(M/R^3)^{1/2}$ in order to include in an appropriate way the rotational corrections. In this case the prior knowledge of the rotation rate of the star may be needed. Potentially, this information can be deduced from gravitational waves due to the r-mode instability. In this case the frequency is directly proportional to the rotation rate of the star, cf. equation (21), and the r-mode instability signal is expected to be much stronger than those of the other modes. Hence, one can immediately infer the rotation of the star. The rotation rate will obviously change due to the r-mode, but for time intervals of the order of 1-2 secs (the typical damping time of other modes) it can be considered as constant. The information about the rotation rate can then be combined with empirical formulas that can be derived for the modes of rotating stars in order to extract all of the parameters of the star.

Strong magnetic fields can also affect the mode frequencies of the neutron stars. Preliminary results for Newtonian stars have shown that typical magnetic fields of the order of $10^{11} - 10^{14}$ Gauss, do not produce significant shifts in the mode frequencies. One must to consider extremely strong magnetic fields, like in magnetars ($10^{15} - 10^{16}$ G) in order to get a significant frequency shift. In fact, this highlights an interesting issue for the magnetars: if one can detect gravitational waves from the starquakes induced by their strong magnetic fields, a possible shift in the spectrum may provide a measure of the magnetic field strength.

Finally, it is worth mentioning that a detection mode pulsation modes from old, cold neutron stars could also provide unique insights into the superfluid nature of neutron star cores. It is generally believed that once a neutron star cools below a few times 10^9 K (a few months after its birth) the bulk of its core will become superfluid. Thus, the more than 1000 observed pulsars provide useful laboratories for studying large scale superfluidity. In the simplest description, a superfluid neutron star core can be discussed in terms of two distinct fluids. One of these fluids represents the superfluid neutrons and the other fluid represents all charged components (which are expected to be coupled on a relatively short timescale). The fact that these two fluids — the "neutrons" and the "protons" — can flow more or less independently provides one of the main distinguishing dynamical features of a superfluid neutron star. In particular, one can show that two sets of pulsation modes are interlaced in the spectrum of a superfluid neutron star core [56, 57]. One set of modes are the familiar p-modes, for which the two fluids tend to move together. The other

set of modes are distinguished by the fact that the protons and neutrons are largely "countermoving". This class of modes is unique to the two-fluid system. Of particular interest for our current discussion is the fact that the "superfluid mode" frequencies are (locally) approximated by

$$\omega_s^2 \approx \frac{m_p}{m_p^*}\frac{l(l+1)}{r^2}c_p^2 \ , \tag{24}$$

where c_p^2 is (roughly) the sound speed in the proton fluid, r is the radial coordinate, and l is the index of the relevant spherical harmonic $Y_{lm}(\theta,\varphi)$ used to describe the angular dependency of the mode. From this relation it is clear that an observation of these modes would provide potentially unique information regarding the nature of large scale superfluidity, and could put useful constraints on crucial parameters such as the ratio between the "bare" and "effective" proton masses m_p and m_p^* (estimates to lie in the range $0.3 \le m_p^*/m_p \le 0.8$). We think this is a very exciting prospect that should motivate future efforts in this field.

Acknowledgements

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